Write the answers to each ***Group*** *in a separate answer-book.*

2024

MATHEMATICS — MDC

Paper : CC-1

(Calculus, Geometry and Vector Analysis)

Full Marks : 75

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Symbols have their usual meanings.

Group-A

[Calculus]

(Marks : 20)

1. Answer ***any four*** questions : 2×4

(a) If $p^2 = a^2\cos^2\theta + b^2\sin^2\theta$, prove that $p + \dfrac{d^2p}{d\theta^2} = \dfrac{a^2b^2}{p^3}$.

(b) Calculate the total length of the cardioide $r = a(1 + \cos\theta)$.

(c) Find $\dfrac{dy}{dx}$, where $(\cos x)^y = (\sin y)^x$.

(d) Find the n-th derivative of $y = \dfrac{a-x}{a+x}$.

(e) If $I_n = \int_0^1 (1-x^2)^n\, dx$, then prove that $(2n + 1)I_n = 2nI_{n-1}$.

(f) Find $\lim\limits_{h\to 0} \dfrac{f(a+h)+f(a-h)-2f(a)}{h^2}$ provided $f'''(x)$ exists for all x.

(g) If $f(x) = x\sin\dfrac{1}{x}, x \neq 0$ and $f(0) = 0$, show that $f'(0)$ does not exist.

2. Answer ***any three*** questions :

(a) For all $x > 0$, prove that $\dfrac{x}{1+x} < \log(1+x) < x$. 4

(b) Find the area in the first quadrant included between the parabola $y^2 = bx$ and the circle $x^2 + y^2 = 2bx$. Find also when $b = 4$. 4

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(c) If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, for $|x| < 1$, show that $(1 - x^2)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2 y_n = 0$. 4

(d) If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$, show that $I_{n+1} - I_{n-1} = \frac{1}{n}$.

Using this relation, find the value of $\int_0^{\frac{\pi}{4}} \tan^8 x \, dx$. 2+2

(e) Consider the straight line passing through origin O and a point $P(h, r)$. The portion OP is revolved about x-axis. Show that the surface area of the cone thus generated is πrl, where l is the slant height of the cone. Also show that the volume of the cone is $\frac{1}{3}\pi r^2 h$. 4

(f) Find the value of a and b if $\lim_{x \to 0} \frac{a \sin x - bx \cos x}{x^3} = \frac{1}{3}$. 4

Group - B

[Geometry]

(Marks : 35)

3. Answer ***any two*** questions :

(a) The coordinate axes are rotated through an angle $\frac{\pi}{4}$. If the transformed coordinates of a point are $(0, \sqrt{2})$, find the original coordinates. 2½

(b) Find the equation of the sphere through the circle $x^2 + y^2 + z^2 = 9$, $x + y - 2z - 4 = 0$ and the origin. 2½

(c) Determine the nature of the conic $r = \frac{1}{4 - 5\cos\theta}$. Find the eccentricity and the length of the latus rectum. ½+1+1

(d) Find the equation to the generating lines of the paraboloid $(x + y + z)(2x + y - z) = 6z$, which pass through the point (1, 1, 1). 2½

4. Answer ***any five*** questions : 6×5

(a) The tangents at two points P and Q of a parabola whose focus is S, meet at T. Show that $SP \cdot SQ = ST^2$.

(b) Reduce the equation $7x^2 - 6xy - y^2 + 4x - 4y - 2 = 0$ to its canonical form and find the nature of the conic.

(c) Find the equations of the generators of the hyperbola $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, passing through the point $(a\cos\alpha,\ b\sin\alpha,\ 0)$.

(d) If the straight line $r\cos(\theta - \alpha) = p$ touches the conic $\frac{l}{r} = 1 + e\cos\theta$, then prove that $(l\cos\alpha - ep)^2 + l^2\sin^2\alpha = p^2$.

(e) Find the equation of the right circular cylinder of radius 3 with axis passing through (1, 3, 4) and have 1, –2, 3 as its direction ratios.

(f) Find the nature of the surface given by the equation $2x^2 + 3y^2 - 8x + 6y - 12z + 11 = 0$.

(g) If PSP' be a focal chord of a conic, then show that $\frac{1}{SP} + \frac{1}{SP'} = \frac{2}{l}$, where l is the semi-latus rectum.

(h) Find the centre and the radius of the circle : $x^2 + y^2 + z^2 - 2x + 4y + 6z - 2 = 0,\ x + 3y + 2z = 15$.

(i) Show that the foot of the perpendicular from the focus to any tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ lies on the circle $x^2 + y^2 = a^2$.

Group - C

[Vector Analysis]

(Marks : 20)

5. Answer ***any four*** questions : 2×4

(a) Prove that, $\vec{a} \times (\vec{b} \times \vec{c})$, $\vec{b} \times (\vec{c} \times \vec{a})$ and $\vec{c} \times (\vec{a} \times \vec{b})$ are coplanar.

(b) If $\vec{r} = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\,\hat{k}$, show that, $\frac{d\vec{r}}{dt} \cdot \frac{d^2\vec{r}}{dt^2} \times \frac{d^3\vec{r}}{dt^3} = a^2 b$.

(c) Find the vector equation of the straight line passing through the point (–1, 4, 3) and parallel to the vector $4\hat{i} + 3\hat{j} + 2\hat{k}$.

(d) A force $\vec{F} = 3\hat{i} + 2\hat{j} - 4\hat{k}$ is applied at the point (1, –1, 2). Find the moment of $\vec{F}$ about the point (2, – 1, 3).

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(e) Evaluate $\int_2^3 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt$, where $\vec{r} = t^3\hat{i} + 2t^2\hat{j} + 3t\hat{k}$.

(f) If a particle is acted on by constant forces $\left(4\vec{i} + \vec{j} - 3\vec{k}\right)$ and $\left(3\vec{i} + \vec{j} - \vec{k}\right)$ and is displaced from the point $\left(\vec{i} + 2\vec{j} + 3\vec{k}\right)$ to the point $\left(5\vec{i} + 4\vec{j} - \vec{k}\right)$, find the total work done by the forces.

(g) Solve : $p\vec{x} + \vec{x}\left(\vec{x}.\vec{b}\right) = \vec{a} \times \vec{b} + \vec{c}$.

6. Answer ***any three*** questions : 4×3

(a) If $\vec{F} = x^2y\hat{i} + xy^2z\hat{j} + y^3z^2\hat{k}$ and $\vec{G} = x\hat{i} - yz^2\hat{j} + xyz\hat{k}$, then show that $\frac{\partial^2}{\partial x \partial y}\left(\vec{F}.\vec{G}\right) = \frac{\partial^2}{\partial y \partial x}\left(\vec{F}.\vec{G}\right)$ and find $\frac{\partial^2}{\partial x \partial y}\left(\phi\vec{F}\right)$ at $(-1,-1,1)$, where $\phi = xyz$.

(b) If $\phi = 3x^2yz$ and C is the curve $x = t^2$, $y = t^3$, $z = t$ from $t = 0$ to $t = 1$, evaluate the vector line integral $\int_C \phi\, d\vec{r}$.

(c) Find the vector equation of the plane passing through the point $\left(8\hat{i} + 2\hat{j} - 3\hat{k}\right)$ and perpendicular to each of the planes $\vec{r}.\left(2\hat{i} - \hat{j} + 2\hat{k}\right) = 0$ and $\vec{r}.\left(\hat{i} + 3\hat{j} - 5\hat{k}\right) + 5 = 0$.

(d) A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$. Find the components of velocity and acceleration at time $t = 1$, in the direction of $\hat{i} - 3\hat{j} + 2\hat{k}$.

(e) Show that $\vec{a} \times \vec{b}$, $\vec{b} \times \vec{c}$, $\vec{c} \times \vec{a}$ are coplanar if and only if $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

(f) If $\vec{r} = (a\cos t)\vec{i} + (a\sin t)\vec{j} + (at\tan\alpha)\vec{k}$, evaluate $\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right]$.
